

The Cantor set.

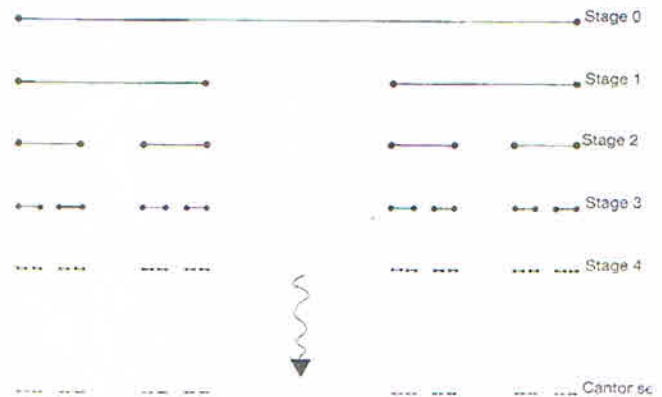
(This is a conversation between Henry and Anne-Lida.)

- Henry : Georg Cantor was a German mathematician who invented a very curious set in about 1883 [...]To get a Cantor set you start with a line segment of length 1, and remove its middle third. Now remove the middle third of each remaining piece. Repeat, forever. What is left is the Cantor set. (Figure 1.6)
- Anne-Lida : I don't see how there can be *anything* left, Henry.
- Henry :Oh, but there is. All the end-points of all the smaller segments are left, for a start. And many others. But you are right in one way, my dear. What is the length of the Cantor set ?
- Anne-Lida :Its ends are distance 1 apart, Henry.
- Henry :No, I meant the length not counting the gaps.
- Anne-Lida :I have no idea, Henry. But it looks very small to me. The set is mostly holes.
- [...]
- Henry :The length reduces to 2/3 the size at each stage, so the total length after the n th stage is $(\frac{2}{3})^n$. As n tends to infinity, this tends to 0. The length of the Cantor set is zero.

Ian Stewart, « Game, Set & Math, enigmas and conundrums », Dover publications, 2007.

Stage	Segment length	Number of segments	Total set length	Number of end-points
0	1	1	1	2
1	$\frac{1}{3}$	2	$2 \times \frac{1}{3} = \frac{2}{3}$	2^2
2				
3				
4				

Table 1



1.6 Construction of the Cantor set by removing middle thirds. Its length is zero, but contains infinitely many points.

TASKS :

- 1) Fill in table 1 above with integers or fractions raised to the appropriate power.
- 2) Conjecture the line of table 1 at stage n .
- 3) Explain why the Cantor set at stage n contains more than $2^{(n+1)}$ points.
- 4) Explain why the Cantor set at stage n has length $(\frac{2}{3})^n$.
- 5) Let n go to infinity and give the length of the Cantor set. Justify your answer.
- 6) How many points are contained in this Cantor set ?
- 7) Using the preceding results, do you think it is possible for a set to be simultaneously of length zero and non empty ? Explain your reasoning.