# The Binomial Distribution

In many cases, it is appropriate to summarize a group of independent observations by the number of observations in the group that represent one of two outcomes. For example, the proportion of individuals in a random sample who support one of two political candidates fits this description. In this case, the statistic $\hat{p}$ is the *count* *X* of voters who support the candidate divided by the total number of individuals in the group *n*. This provides an estimate of theparameter *p*, the proportion of individuals who support the candidate in the entire population.

The ***binomial distribution*** describes the behavior of a count variable *X* if the following conditions apply:

> ***1:*** *The number of observations n is fixed.*
> ***2:*** *Each observation is independent.*
> ***3:*** *Each observation represents one of two outcomes ("success" or "failure").*
> ***4:*** *The probability of "success" p is the same for each outcome.*

If these conditions are met, then *X* has a binomial distribution with parameters *n* and *p*, abbreviated *B(n,p)*.

Example

Suppose individuals with a certain gene have a 0.70 probability of eventually contracting a certain disease. If 100 individuals with the gene participate in a lifetime study, then the distribution of the random variable describing the number of individuals who will contract the disease is distributed *B(100,0.7)*.

***Note: The sampling distribution of a count variable is only well-described by the binomial distribution is cases where the population size is significantly larger than the sample size. As a general rule, the binomial distribution should not be applied to observations from a simple random sample (SRS) unless the population size is at least 10 times larger than the sample size.***

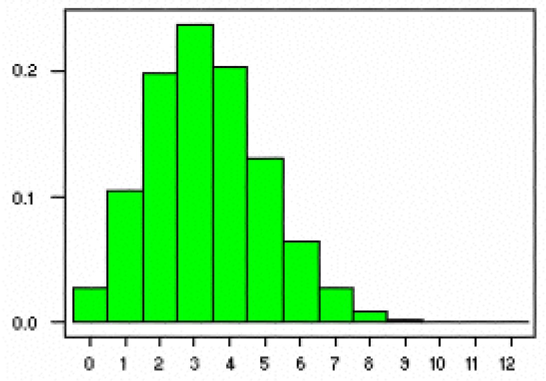
To find probabilities from a binomial distribution, one may either calculate them directly, use a binomial table, or use a computer. The number of sixes rolled by a single die in 20 rolls has a *B(20,1/6)* distribution. The probability of rolling more than 2 sixes in 20 rolls, $P(X>2)$, is equal to $1 - P(X\leq 2) = 1 - (P(X=0) + P(X=1) + P(X=2))$. Using the MINITAB command "cdf" with subcommand "binomial n=20 p=0.166667" gives the cumulative distribution function as follows:

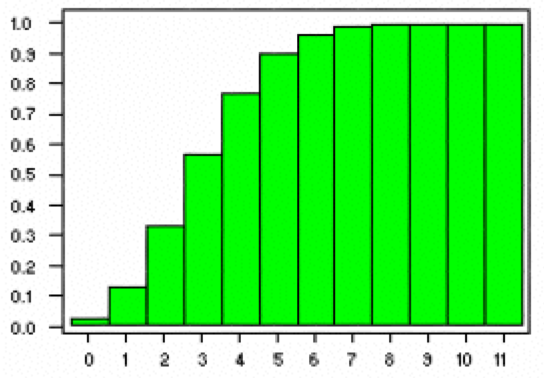
```
Binomial with n = 20 and p = 0.166667

         x      P( X <= x)
         0         0.0261
         1         0.1304
         2         0.3287
         3         0.5665
         4         0.7687
         5         0.8982
         6         0.9629
         7         0.9887
         8         0.9972
```

9 0.9994

The corresponding graphs for the probability density function and cumulative distribution function for the *B(20,1/6)* distribution are shown below:




Since the probability of 2 or fewer sixes is equal to 0.3287, the probability of rolling more than 2 sixes = 1 - 0.3287 = 0.6713.

**The probability that a random variable *X* with binomial distribution *B(n,p)* is equal to the value *k*, where *k = 0, 1,....,n* , is given by**

$$P(X=k)=\binom{n}{k}p^{k}(1-p)^{n-k}$$

, where

$$\binom{n}{k}=\frac{n!}{k!(n-k)!}$$

.

The latter expression is known as the ***binomial coefficient***, stated as "*n choose k*," or the number of possible ways to choose *k* "successes" from *n* observations. For example, the number of ways to achieve 2 heads in a set of four tosses is "4 choose 2", or 4!/2!2! = (4*3)/(2*1) = 6. The possibilities are {HHTT, HTHT, HTTH, TTHH, THHT, THTH}, where "H" represents a head and "T" represents a tail. The binomial coefficient multiplies the probability of *one* of these possibilities (which is $(1/2)^2(1/2)^2$ = 1/16 for a fair coin) by the number of ways the outcome may be achieved, for a total probability of 6/16.

**Mean and Variance of the Binomial Distribution**

The binomial distribution for a random variable *X* with parameters *n* and *p* represents the sum of *n* independent variables *Z* which may assume the values 0 or 1. If the probability that each *Z* variable assumes the value 1 is equal to *p*, then the mean of each variable is equal to *1*p + 0*(1-p) = p*, and the variance is equal to *p(1-p)*. By the addition properties for independent random variables, the mean and variance of the binomial distribution are equal to the sum of the means and variances of the *n* independent *Z* variables, so

$$\mu_X = np$$

$$\sigma_X^2 = np(1-p)$$

These definitions are intuitively logical. Imagine, for example, 8 flips of a coin. If the coin is fair, then $p = 0.5$. One would expect the mean number of heads to be half the flips, or $np = 8*0.5 = 4$. The variance is equal to $np(1-p) = 8*0.5*0.5 = 2$.

**Sample Proportions**

If we know that the count *X* of "successes" in a group of *n* observations with success probability *p* has a binomial distribution with mean *np* and variance *np(1-p)*, then we are able to derive information about the distribution of the ***sample proportion*** $\hat{p}$ , the count of successes *X* divided by the number of observations *n*. By the multiplicative properties of the mean, the mean of the distribution of *X/n* is equal to the mean of *X* divided by *n*, or $np/n = p$. This proves that the sample proportion $\hat{p}$ is an *unbiased estimator* of the population proportion *p*. The variance of *X/n* is equal to the variance of *X* divided by $n^2$, or $(np(1-p))/n^2 = (p(1-p))/n$ . This formula indicates that as the size of the sample increases, the variance decreases.

In the example of rolling a six-sided die 20 times, the probability *p* of rolling a six on any roll is 1/6, and the count *X* of sixes has a *B(20, 1/6)* distribution. The mean of this distribution is 20/6 = 3.33, and the variance is 20*1/6*5/6 = 100/36 = 2.78. The mean of the *proportion* of sixes in the 20 rolls, *X/20*, is equal to $p = 1/6 = 0.167$, and the variance of the proportion is equal to (1/6*5/6)/20 = 0.007.

**Normal Approximations for Counts and Proportions**

**For large values of *n*, the distributions of the count *X* and the sample proportion $\hat{p}$ are approximately [normal](). This result follows from the [Central Limit Theorem](). The mean and variance for the approximately normal distribution of *X* are *np* and *np(1-p)*, identical to the mean and variance of the binomial(*n,p*) distribution. Similarly, the mean and variance for the approximately normal distribution of the sample proportion are *p* and *(p(1-p)/n)*.**

*Note: Because the normal approximation is not accurate for small values of n, a good rule of thumb is to use the normal approximation only if* $np \geq 10$ *and* $np(1-p) \geq 10$.

For example, consider a population of voters in a given state. The true proportion of voters who favor candidate A is equal to 0.40. Given a sample of 200 voters, what is the probability that more than half of the voters support candidate A?

The count *X* of voters in the sample of 200 who support candidate A is distributed *B(200,0.4)*. The mean of the distribution is equal to 200*0.4 = 80, and the variance is equal to 200*0.4*0.6 = 48. The standard deviation is the square root of the variance, 6.93. The probability that more than half of the voters in the sample support candidate A is equal to the probability that *X* is greater than 100, which is equal to $1- P(X \leq 100)$.

To use the normal approximation to calculate this probability, we should first acknowledge that the normal distribution is *continuous* and apply the ***continuity correction***. This means that the probability for a single discrete value, such as 100, is extended to the probability of the *interval* (99.5,100.5). Because we are interested in the probability that *X* is less than or equal to 100, the normal approximation applies to the upper limit of the interval, 100.5. If we were interested in the probability that *X* is strictly less than 100, then we would apply the normal approximation to the lower end of the interval, 99.5.

So, applying the continuity correction and standardizing the variable *X* gives the following:
$1 - P(X \leq 100)$
$= 1 - P(X \leq 100.5)$
$= 1 - P(Z \leq (100.5 - 80)/6.93)$
$= 1 - P(Z \leq 20.5/6.93)$
$= 1 - P(Z \leq 2.96) = 1 - (0.9985) = 0.0015$. Since the value 100 is nearly three standard deviations away from the mean 80, the probability of observing a count this high is extremely small.

Site : http://www.stat.yale.edu/Courses/1997-98/101/binom.htm