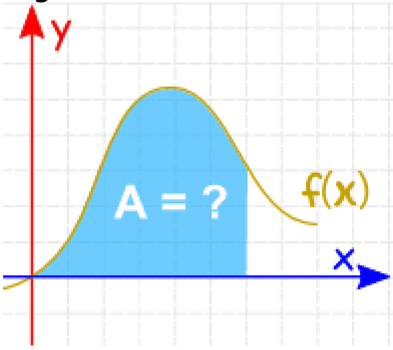
Integration, Definite integrals, Antiderivative/primitive function : examples and definitions

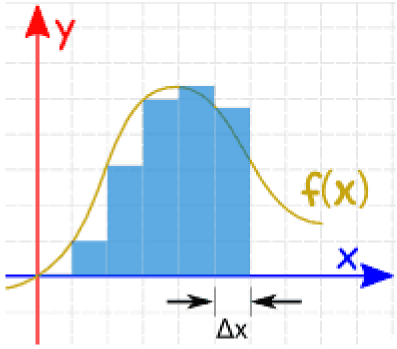
Integration is a way of adding slices to find the whole (extracted from *Mathsisfun.com* website)
Integration can be used to find areas, volumes, central points and many useful things.
But it is easiest to start with finding the **area under the curve of a function** like this:



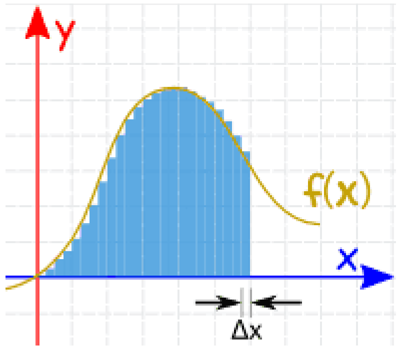
What is the area under **y = f(x)** ?

Slices

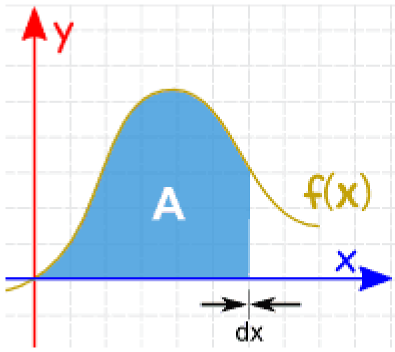
We could calculate the function at a few points and **add up slices of width Δx** like this (but the answer won't be very accurate):



We can make **Δx** a lot smaller and **add up many small slices** (answer is getting better):



And as the slices **approach zero in width**, the answer approaches the**true answer**.
We now write **dx** to mean the **Δx** slices are approaching zero in width.



That is a lot of adding up!

But we don't have to add them up, as there is a "shortcut". Because ...

... finding an Integral is the **reverse** of finding a Derivative.

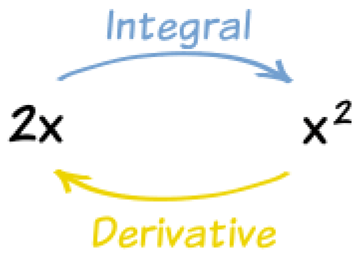
(So you should really know about Derivatives before reading more!)

Like here:

Example: What is an integral of 2x?

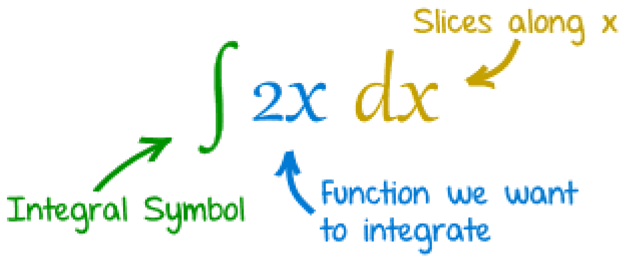
We know that the derivative of x^2 is $2x$.

... so an integral of $2x$ is x^2



Notation

The symbol for "Integral" is a stylish "S"
(for "Sum", the idea of summing slices):



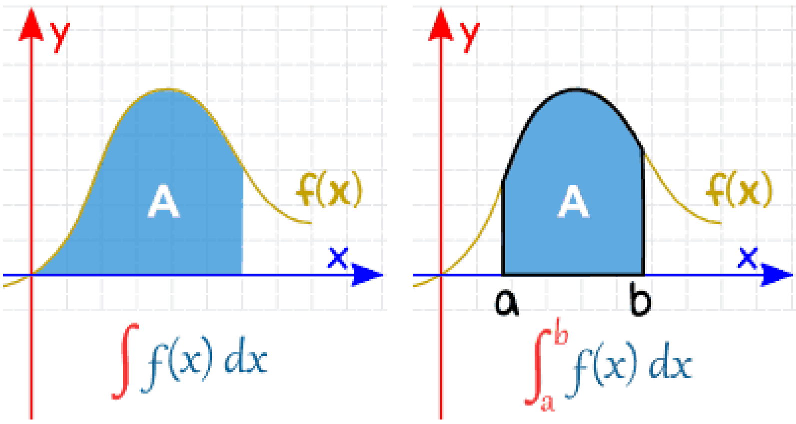
After the Integral Symbol we put the function we want to find the integral of (called the Integrand),
and then finish with **dx** to mean the slices go in the x direction (and approach zero in width).
And here is how we write the answer:

$$\int 2x\,dx = x^2 + C$$

Definite Integral

A **Definite Integral** has start and end values: in other words there is an **interval** (a to b).

The values are put at the bottom and top of the "S", like this:



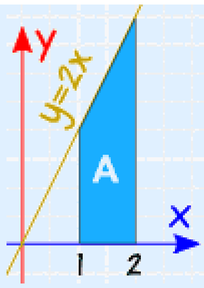
Indefinite Integral
(no specific values)

Definite Integral
(from **a** to **b**)

Example:

The **Definite Integral**, from 1 to 2, of **2x** dx:

$$\int_1^2 2x\,dx$$



The ***Indefinite* Integral** is: $\int 2x\,dx = x^2 + C$

- At x=1: $\int 2x\,dx = 1^2 + C$
- At x=2: $\int 2x\,dx = 2^2 + C$

Subtract:

→ $(2^2 + C) - (1^2 + C)$

→ $2^2 + C - 1^2 - C$

→ $4 - 1 + C - C = 3$

And "C" gets cancelled out ... so with Definite Integrals we can ignore C.

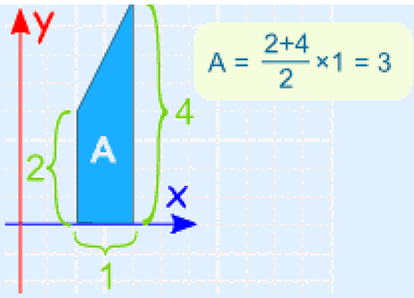
In fact we can give the answer directly like this:

$$\int_1^2 2x\,dx = 2^2 - 1^2 = 3$$

We can check that, by calculating the area of the shape:

Yes, it has an area of 3.

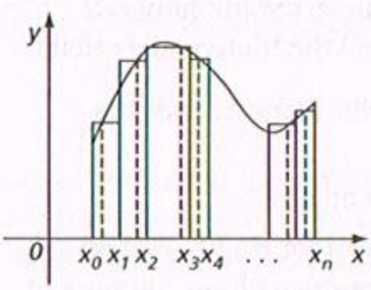
(Yay!)



integral Let f be a function defined on the closed interval $[a, b]$. Take points $x_0, x_1, x_2, \ldots, x_n$ such that $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$, and in each subinterval $[x_i, x_{i+1}]$ take a point c_i. Form the sum

$$\sum_{i=0}^{n-1} f(c_i)(x_{i+1} - x_i);$$

that is, $f(c_0)(x_1 - x_0) + f(c_1)(x_2 - x_1) + \ldots + f(c_{n-1})(x_n - x_{n-1})$. Such a sum is called a Riemann sum for f over $[a, b]$. Geometrically, it gives the sum of the areas of n rectangles, and is an approximation to the area under the curve $y = f(x)$ between $x = a$ and $x = b$.



The (Riemann) integral of f over $[a, b]$ is defined to be the limit I (in a sense that needs more clarification than can be given here) of such a

Riemann sum as n, the number of points, increases and the size of the subintervals gets smaller. The value of I is denoted by

$$\int_a^b f(x)\,dx, \quad \text{or} \quad \int_a^b f(t)\,dt,$$

where it is immaterial what letter, such as x or t, is used in the integral. The intention is that the value of the integral is equal to what is intuitively understood to be the area under the curve $y=f(x)$. Such a limit does not always exist, but it can be proved that it does if, for example, f is a *continuous function on $[a, b]$.

If f is continuous on $[a, b]$ and F is defined by

$$F(x) = \int_a^x f(t)\,dt,$$

then $F'(x)=f(x)$ for all x in $[a, b]$, so that F is an *antiderivative of f. Moreover, if an antiderivative ϕ of f is known, the integral

$$\int_a^b f(t)\,dt$$

can be easily evaluated: the *Fundamental Theorem of Calculus gives its value as $\phi(b) - \phi(a)$. Of the two integrals

$$\int_a^b f(x)\,dx \quad \text{and} \quad \int f(x)\,dx,$$

the first, with limits, is called a definite integral and the second, which denotes an antiderivative of f, is an indefinite integral.

F is called **an antiderivative, primitive function, primitive integral** or **an indefinite integral** of the function $\boldsymbol{f}$. F is a differentiable function whose derivative is equal to the original function f.

antiderivative Given a *real function f, any function ϕ such that $\phi'(x)=f(x)$, for all x (in the domain of f), is an antiderivative of f. If ϕ_1 and ϕ_2 are both antiderivatives of a *continuous function f, then $\phi_1(x)$ and $\phi_2(x)$ differ by a constant. In that case, the notation

$$\int f(x)\,dx$$

may be used for an antiderivative of f, with the understanding that an arbitrary constant can be added to any antiderivative. Thus,

$$\int f(x)\,dx + c,$$

where c is an arbitrary constant, is an expression that gives all the antiderivatives.